Fall 2021 AMC 12A SOLUTIONS

Stevenson Math Team*

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Contents

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0 Problems

ANSWER KEY: CEBEB DBDED ECAED BBEBD CDADE

Problem 1.

What is the value of $\dfrac{(2112-2021)^2}{169}$?

(A) 7 **(B)** 21 **(C)** 49 **(D)** 64 **(E)** 91

Problem 2.

Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Problem 3.

Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

(A) $2\frac{3}{4}$ **(B)** $3\frac{3}{4}$ **(C)** $4\frac{1}{2}$ **(D)** $5\frac{1}{2}$ **(E)** $6\frac{3}{4}$

Problem 4.

The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A?

(**A**) 1 (**B**) 3 (**C**) 5 (**D**) 7 (**E**) 9

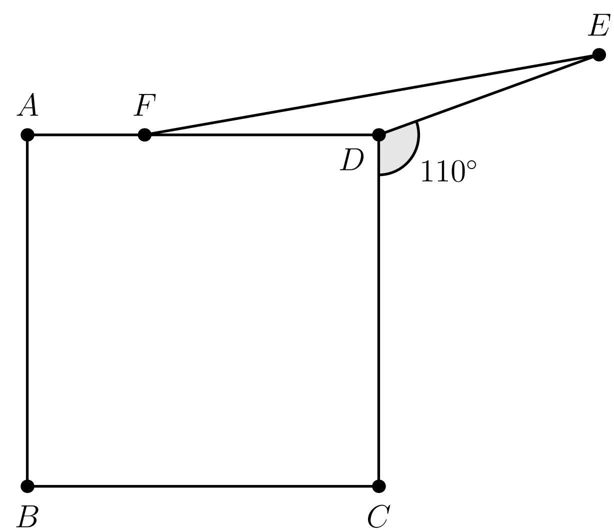
Problem 5.

Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 11 **(E)** 15

Problem 6.

As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110°$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



(A) 160 **(B)** 164 **(C)** 166 **(D)** 170 **(E)** 174

Problem 7.

A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are $50, 20, 20, 5,$ and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?

(A) -18.5 **(B)** -13.5 **(C)** 0 **(D)** 13.5 **(E)** 18.5

Problem 8.

Let M be the least common multiple of all the integers 10 through 30, inclusive. Let N be the least common multiple of M, 32, 33, 34, 35, 36, 37, 38, 39, and 40. What is the value of $\frac{N}{M}$?

(A) 1 **(B)** 2 **(C)** 37 **(D)** 74 **(E)** 2886

Problem 9.

A right rectangular prism whose surface area and volume are numerically equal has edge lengths $\log_2 x$, $\log_3 x$, and $\log_4 x$. What is x?

(A) $2\sqrt{6}$ **(B)** $6\sqrt{6}$ **(C)** 24 **(D)** 48 **(E)** 576

Problem 10.

The base-nine representation of the number N is $27{,}006{,}000{,}052_{\text{nine}}$. What is the remainder when N is divided by 5?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Problem 11.

Consider two concentric circles of radius 17 and 19. The larger circle has a chord, half of which lies inside the smaller circle. What is the length of the chord in the larger circle?

(A) $12\sqrt{2}$ **(B)** $10\sqrt{3}$ **(C)** $\sqrt{17 \cdot 19}$ **(D)** 18 **(E)** $8\sqrt{6}$

Problem 12.

What is the number of terms with rational coefficients among the 1001 terms of the expression $(x\sqrt[3]{2} + y\sqrt{3})^{1000}$?

(A) 0 **(B)** 166 **(C)** 167 **(D)** 500 **(E)** 501

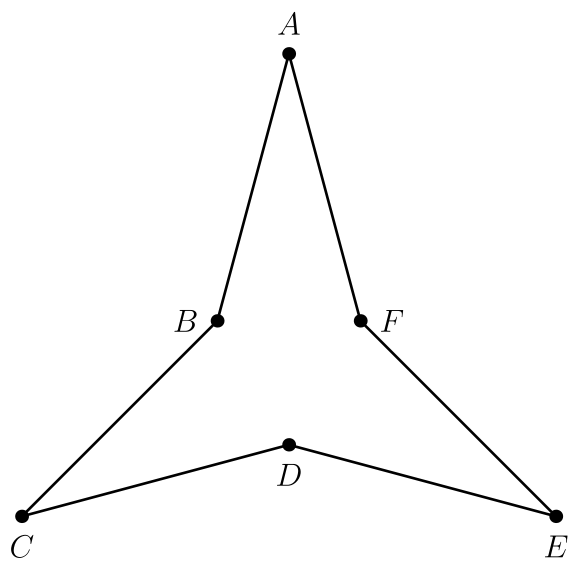
Problem 13.

The angle bisector of the acute angle formed at the origin by the graphs of the lines $y = x$ and $y = 3x$ has equation $y = kx$. What is k?

(A) $\frac{1+\sqrt{5}}{2}$ **(B)** $\frac{1+\sqrt{7}}{2}$ **(C)** $\frac{2+\sqrt{3}}{2}$ **(D)** 2 **(E)** $\frac{2+\sqrt{5}}{2}$

Problem 14.

In the figure, equilateral hexagon $ABCDEF$ has three nonadjacent acute interior angles that each measure 30°. The enclosed area of the hexagon is $6\sqrt{3}$. What is the perimeter of the hexagon?



(A) 4 **(B)** $4\sqrt{3}$ **(C)** 12 **(D)** 18 **(E)** $12\sqrt{3}$

Problem 15.

Recall that the conjugate of the complex number $w = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, is the complex number $\overline{w} = a - bi$. For any complex number z, let $f(z) = 4i\,\overline{z}$. The polynomial $P(z) = z^4 + 4z^3 + 3z^2 + 2z + 1$ has four complex roots: z_1, z_2, z_3, and z_4. Let $Q(z) = z^4 + Az^3 + Bz^2 + Cz + D$ be the polynomial whose roots are $f(z_1)$, $f(z_2)$, $f(z_3)$, and $f(z_4)$, where the coefficients A, B, C, and D are complex numbers. What is $B + D$?

(A) -304 **(B)** -208 **(C)** $12i$ **(D)** 208 **(E)** 304

Problem 16.

An organization has 30 employees, 20 of whom have a brand A computer while the other 10 have a brand B computer. For security, the computers can only be connected to each other and only by cables. The cables can only connect a brand A computer to a brand B computer. Employees can communicate with each other if their computers are directly connected by a cable or by relaying messages through a series of connected computers. Initially, no computer is connected to any other. A technician arbitrarily selects one computer of each brand and installs a cable between them, provided there is not already a cable between that pair. The technician stops once every employee can communicate with each other. What is the maximum possible number of cables used?

(A) 190 **(B)** 191 **(C)** 192 **(D)** 195 **(E)** 196

Problem 17.

How many ordered pairs of positive integers (b, c) exist where both $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ do not have distinct, real solutions?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

Problem 18.

Each of the 20 balls is tossed independently and at random into one of the 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

(A) 1 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 16

Problem 19.

Let x be the least real number greater than 1 such that $\sin(x) = \sin(x^2)$, where the arguments are in degrees. What is x rounded up to the closest integer?

(A) 10 **(B)** 13 **(C)** 14 **(D)** 19 **(E)** 20

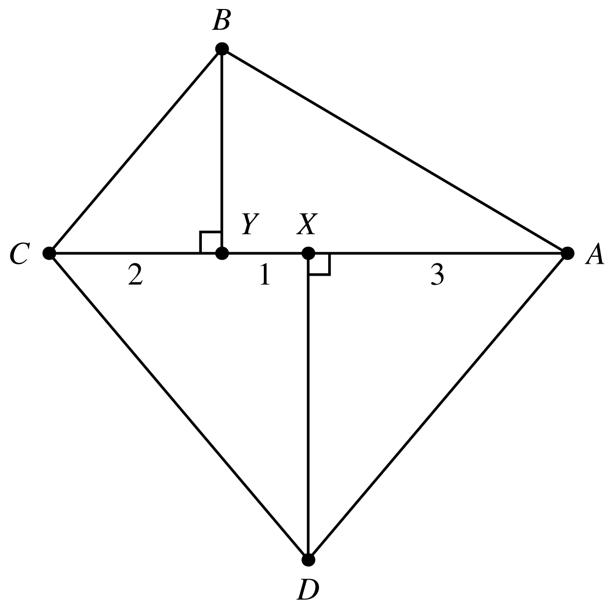
Problem 20.

For each positive integer n, let $f_1(n)$ be twice the number of positive integer divisors of n, and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Problem 21.

Let $ABCD$ be an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$ and $AB = CD$. Points X and Y lie on diagonal $\overline{AC}$ with X between A and Y, as shown in the figure. Suppose $\angle AXD = \angle BYC = 90^\circ$, $AX = 3$, $XY = 1$, and $YC = 2$. What is the area of $ABCD$?



(A) 15 **(B)** $5\sqrt{11}$ **(C)** $3\sqrt{35}$ **(D)** 18 **(E)** $7\sqrt{7}$

Problem 22.

Azar and Carl play a game of tic-tac-toe. Azar places an X in one of the boxes in the 3-by-3 array of boxes, then Carl places an O in one of the remaining boxes. After that, Azar places an X in one of the remaining boxes, and so on until all 9 boxes are filled or one of the players has 3 of their symbols in a row — horizontal, vertical, or diagonal — whichever comes first, in which case that player wins the game. Suppose the players make their moves at random, rather than trying to follow a rational strategy, and that Carl wins the game when he places his third O. How many ways can the board look after the game is over?

(A) 36 **(B)** 112 **(C)** 120 **(D)** 148 **(E)** 160

Problem 23.

A quadratic polynomial $p(x)$ with real coefficients and leading coefficient 1 is called disrespectful if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

(A) $\frac{5}{16}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{8}$ **(D)** 1 **(E)** $\frac{9}{8}$

Problem 24.

Convex quadrilateral $ABCD$ has $AB = 18$, $\angle A = 60^\circ$, and $\overline{AB} \parallel \overline{CD}$. In some order, the lengths of the four sides form an arithmetic progression, and side $\overline{AB}$ is a side of maximum length. The length of another side is a. What is the sum of all possible values of a?

(A) 24 **(B)** 42 **(C)** 60 **(D)** 66 **(E)** 84

Problem 25.

Let $m \geq 5$ be an odd integer, and let $D(m)$ denote the number of quadruples (a_1, a_2, a_3, a_4) of distinct integers with $1 \leq a_i \leq m$ for all i such that m divides $a_1 + a_2 + a_3 + a_4$. There is a polynomial

$$q(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

such that $D(m) = q(m)$ for all odd integers $m \geq 5$. What is c_1?

(**A**) -6 (**B**) -1 (**C**) 4 (**D**) 6 (**E**) 11

1 AMC 12A 2021/1

What is the value of $\frac{(2112-2021)^2}{169}$?

(A) 7 **(B)** 21 **(C)** 49 **(D)** 64 **(E)** 91

Solution. $\boxed{\text{E}}$

$$\frac{7^2 \cdot 13^2}{13^2} = 49.$$

2 AMC 12A 2021/2

Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Solution. $\boxed{\text{E}}$

Originally, she shortens the length of the shorter side of the index card, so that $3 \cdot 6 = 18$ square inches. When she shortens the longer side instead, $4 \cdot 5 = 20$ square inches.

3 AMC 12A 2021/3

Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

(A) $2\frac{3}{4}$ **(B)** $3\frac{3}{4}$ **(C)** $4\frac{1}{2}$ **(D)** $5\frac{1}{2}$ **(E)** $6\frac{3}{4}$

Solution. $\boxed{\text{B}}$

Mr. Lopez's time taken for route A is equal to

$$6 \cdot \frac{60}{30} = 12 \text{ minutes.}$$

His time taken for route B is equal to

$$\left(5 - \frac{1}{2}\right) \cdot \frac{60}{40} + \frac{1}{2} \cdot \frac{60}{20} = 8\frac{1}{4} \text{ minutes.}$$

Therefore, route B is shorter by $3\frac{3}{4}$ minutes.

4 AMC 12A 2021/4

The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A?

(**A**) 1 (**B**) 3 (**C**) 5 (**D**) 7 (**E**) 9

Solution. $\boxed{\text{E}}$

If $A = 1$ or $A = 7$, then the number is divisible by 3. If $A = 5$, then the number is divisible by 5. If $A = 3$, then the number is divisible by 11 (alternating digits sum to the same value). Therefore, A must be 9.

5 AMC 12A 2021/5

Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

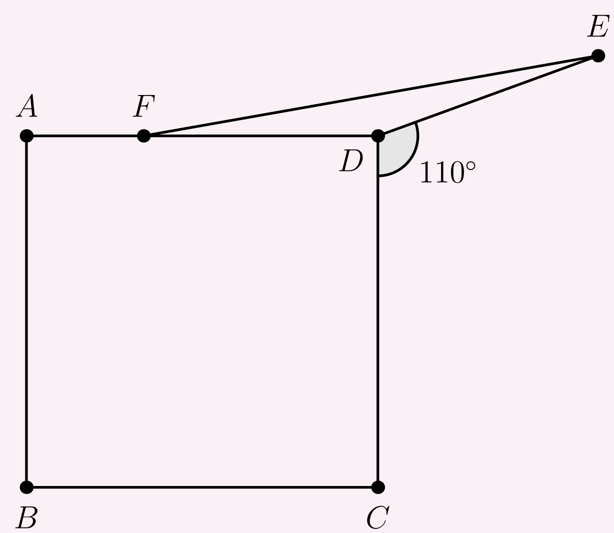
(A) 6 **(B)** 8 **(C)** 10 **(D)** 11 **(E)** 15

Solution. $\boxed{\text{B}}$

Each pole is spaced $\frac{5280}{41-1} = 132$ feet apart. Therefore, Elmer's stride is $132/44 = 3$ feet long, while Oscar's leap is $132/12 = 11$ feet long, making Oscar's leap 8 feet longer.

6 AMC 12A 2021/6

As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110^\circ$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



(**A**) 160 (**B**) 164 (**C**) 166 (**D**) 170 (**E**) 174

Solution. D

$$\angle AFE = 180 - \angle EFD = 180 - \frac{(110-90)}{2} = 170^\circ.$$

7 AMC 12A 2021/7

A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are $50, 20, 20, 5$, and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student was picked at random and the number of students in their class, including the student, is noted. What is $t - s$?

(A) -18.5 **(B)** -13.5 **(C)** 0 **(D)** 13.5 **(E)** 18.5

Solution. $\boxed{\text{B}}$

t is equivalent to the average of the sizes of each class:

$$\frac{1}{5}(50 + 20 + 20 + 5 + 5)) = 20.$$

s is equivalent to the sum of the sizes squared, divided by the total number of students:

$$\frac{1}{100}\left(50^2 + 20^2 + 20^2 + 5^2 + 5^2\right) = 33.5.$$

Therefore, $t - s = -13.5$.

8 AMC 12A 2021/8

Let M be the least common multiple of all the integers 10 through 30, inclusive. Let N be the least common multiple of M, 32, 33, 34, 35, 36, 37, 38, 39, and 40. What is the value of $\frac{N}{M}$?

(A) 1 **(B)** 2 **(C)** 37 **(D)** 74 **(E)** 2886

Solution. $\boxed{\text{D}}$

Recall that the least common multiple of a set of numbers is equal to the product of each prime factor present raised to the highest exponent as it appears in any of the numbers in the set. Therefore, to calculate $\frac{N}{M}$, it suffices to check which of the numbers 32 through 40 introduce new prime factors.

$32 = 2^5$ introduces a factor of 2, because previously $16 = 2^4$ was the highest exponentiation of 2 present in M.

$33, 34, 35, 36$ do not introduce any new prime factors, because $2^2 = 4, 3^2 = 9, 5, 7, 11$ are all present in M.

37 introduces another prime factor (itself), as it is prime. $38, 39, 40$ introduce no new factors.

Finally, our answer is $2 \cdot 37 = 74$.

9 AMC 12A 2021/9

A right rectangular prism whose surface area and volume are numerically equal has edge lengths $\log_2 x$, $\log_3 x$, and $\log_4 x$. What is x?

(A) $2\sqrt{6}$ **(B)** $6\sqrt{6}$ **(C)** 24 **(D)** 48 **(E)** 576

Solution. $\boxed{\text{E}}$

By virtue of the change-of-base formula for logs, we have:

$$\begin{aligned}(\log_2 x)(\log_3 x)(\log_4 x) &= 2(\log_2 x \cdot \log_3 x + \log_3 x \cdot \log_4 x + \log_4 x \cdot \log_2 x)\\ \implies \frac{(\log x)^3}{\log 2 \log 3 \log 4} &= 2\frac{(\log x)^2}{\log 2 \log 3 + \log 3 \log 4 + \log 4 \log 2}\\ \implies \log x &= 2(\log 4 + \log 2 + \log 3) = \log 24^2\\ \implies x &= 576.\end{aligned}$$

10 AMC 12A 2021/10

The base-nine representation of the number N is $27{,}006{,}000{,}052_{\text{nine}}$. What is the remainder when N is divided by 5?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution. $\boxed{\text{D}}$

Noting that $9 \equiv -1 \pmod 5$, we have:

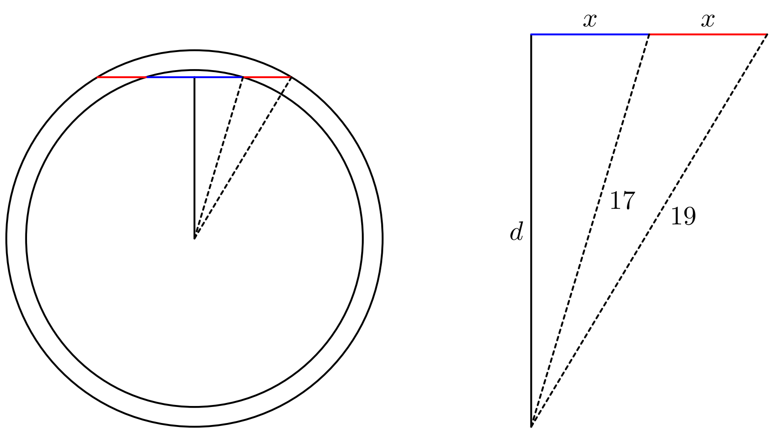
$$\begin{aligned} N &= 2 \cdot 9^{10} + 7 \cdot 9^{9} + 6 \cdot 9^{6} + 5 \cdot 9^{1} + 2 \\ &\equiv 2 - 7 + 6 - 5 + 2 \equiv 3 \pmod 5. \end{aligned}$$

11 AMC 12A 2021/11

Consider two concentric circles of radius 17 and 19. The larger circle has a chord, half of which lies inside the smaller circle. What is the length of the chord in the larger circle?

(A) $12\sqrt{2}$ **(B)** $10\sqrt{3}$ **(C)** $\sqrt{17\cdot 19}$ **(D)** 18 **(E)** $8\sqrt{6}$

Solution. E



As labeled in the above diagrams, let the desired length of our chord be $4x$. Then:

$$d^2 = 19^2 - 4x^2 = 17^2 - x^2,$$

so that

$$x = 2\sqrt{6} \implies 4x = 8\sqrt{6}.$$

12 AMC 12A 2021/12

What is the number of terms with rational coefficients among the 1001 terms of the expression $(x\sqrt[3]{2}+y\sqrt{3})^{1000}$?

(A) 0 **(B)** 166 **(C)** 167 **(D)** 500 **(E)** 501

Solution. $\boxed{\text{C}}$

For each $0 \le k \le 1000$, the coefficient corresponding to the term $x^k y^{1000-k}$ is equal to

$$2^{k/3} \cdot 3^{(1000-k)/2} \cdot \binom{1000}{k}.$$

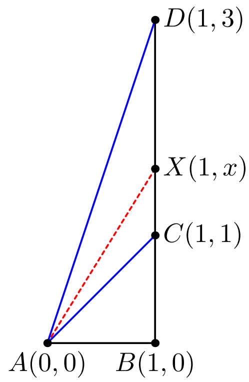
In order for this expression to be rational, we require that each exponent is an integer, implying that we want k even and divisible by 3. In other words, we seek multiples of 6 in the range from 0 to 1000 inclusive, of which there are $\lceil 1000/6 \rceil = 167$.

13 AMC 12A 2021/13

The angle bisector of the acute angle formed at the origin by the graphs of the lines $y = x$ and $y = 3x$ has equation $y = kx$. What is k?

(A) $\frac{1+\sqrt{5}}{2}$ **(B)** $\frac{1+\sqrt{7}}{2}$ **(C)** $\frac{2+\sqrt{3}}{2}$ **(D)** 2 **(E)** $\frac{2+\sqrt{5}}{2}$

Solution. A



Consider the diagram formed by the intersections of the two graphs with the origin and the line $x = 1$. In the notation depiction above, we know that $AD = \sqrt{10}$, $AC = \sqrt{2}$, and we seek AX as our desired angle bisector.

By the angle bisector theorem,

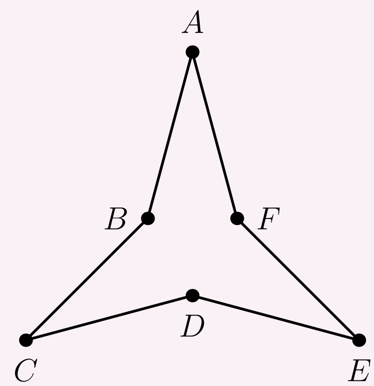
$$CX = \frac{AC}{AC + AD} \cdot CD = \frac{2}{1 + \sqrt{5}}.$$

It follows that the height of X, which is equal to k, is

$$1 + \frac{2}{1 + \sqrt{5}} = \frac{1 + \sqrt{5}}{2}.$$

14 AMC 12A 2021/14

In the figure, equilateral hexagon *ABCDEF* has three nonadjacent acute interior angles that each measure 30°. The enclosed area of the hexagon is $6\sqrt{3}$. What is the perimeter of the hexagon?



(A) 4 **(B)** $4\sqrt{3}$ **(C)** 12 **(D)** 18 **(E)** $12\sqrt{3}$

Solution. $\boxed{\text{E}}$

Let x be the length of each side. The hexagon can be deconstructed into three isosceles triangles and equilateral triangle $\triangle BFD$. Each isosceles triangle has area $\frac{1}{2}x^2 \sin 30^\circ = \frac{x^2}{4}$, while the side length s of the equilateral triangle can be computed with law of cosines:

$$s^2 = x^2 + x^2 - 2x^2 \cos \frac{\sqrt{3}}{2} = \left(2 - \sqrt{3}\right) x^2.$$

By the given area condition, we therefore have that

$$6\sqrt{3} = \frac{3x^2}{4} + \frac{\sqrt{3}}{4}\left(2 - \sqrt{3}\right) x^2,$$

implying that $x = 2\sqrt{3}$.

Remark. Let $X = DE \cap CB$, $Y = CB \cap AF$, and $Z = AF \cap ED$. Using equilateral $\triangle XYZ$, areas can be broken down without having to use the law of cosines – see if you can show how.

15 AMC 12A 2021/15

Recall that the conjugate of the complex number $w = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, is the complex number $\overline{w} = a - bi$. For any complex number z, let $f(z) = 4i\overline{z}$. The polynomial $P(z) = z^4 + 4z^3 + 3z^2 + 2z + 1$ has four complex roots: z_1, z_2, z_3, and z_4. Let $Q(z) = z^4 + Az^3 + Bz^2 + Cz + D$ be the polynomial whose roots are $f(z_1)$, $f(z_2)$, $f(z_3)$, and $f(z_4)$, where the coefficients A, B, C, and D are complex numbers. What is $B + D$?

(**A**) -304 (**B**) -208 (**C**) $12i$ (**D**) 208 (**E**) 304

Solution. $\boxed{\text{D}}$

Because all of the coefficients in P are real, for each complex z that is a root of P, $\overline{z}$ is also a root. This implies that the set of roots of Q is the same as the set of roots of P but multiplied by the constant $4i$.

By Vieta's, this therefore implies that

$$B + D = 3(4i)^2 + 1(4i)^4 = 208.$$

16 AMC 12A 2021/16

An organization has 30 employees, 20 of whom have a brand A computer while the other 10 have a brand B computer. For security, the computers can only be connected to each other and only by cables. The cables can only connect a brand A computer to a brand B computer. Employees can communicate with each other if their computers are directly connected by a cable or by relaying messages through a series of connected computers. Initially, no computer is connected to any other. A technician arbitrarily selects one computer of each brand and installs a cable between them, provided there is not already a cable between that pair. The technician stops once every employee can communicate with each other. What is the maximum possible number of cables used?

(A) 190 **(B)** 191 **(C)** 192 **(D)** 195 **(E)** 196

Solution. $\boxed{\text{B}}$

Finding the maximum possible number of cables used is equivalent to finding the minimum number of cables required to disconnect at least one employee from the rest of the network, starting from a fully connected network ($10 \cdot 20 = 200$ total cables). Because every employee in this state is connected to at least 10 cables, it follows that we require at least 10 cables to disconnect any one employee. Therefore, it is possible for 190 cables to yield a disconnected system, and impossible for 191 cables to not be disconnected, which gives us our answer.

17 AMC 12A 2021/17

How many ordered pairs of positive integers (b, c) exist where both $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ do not have distinct, real solutions?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

Solution. $\boxed{\text{B}}$

Taking discriminants, the problem statement is equivalent to finding (b, c) which simultaneously satisfy

$$\begin{cases} b^2 - 4c \leq 0 \\ c^2 - 4b \leq 0. \end{cases}$$

When $b = 1$, $c = 1$ and $c = 2$ are the only two solutions. When $b = 2$, $c = 1$ and $c = 2$ are the only two solutions. When $b = 3$, $c = 3$ works, and when $b = 4$, $c = 4$ works, giving us 6 total solutions when $b, c \leq 4$.

To show that no other possibilities work, note that $x^2 - 4x > 0$ for all $x \geq 4$. Therefore, given $b, c > 4$, we must have $b^2 - 4c \geq c^2 - 4c > 0$ when $b \geq c$, and $c^2 - 4b > 0$ otherwise (by the same line of reasoning), which contradicts the above system of equations; therefore, our final answer is 6.

18 AMC 12A 2021/18

Each of the 20 balls is tossed independently and at random into one of the 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

(A) 1 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 16

Solution. $\boxed{\text{E}}$

The ratio of p to q is equal to the ratio of the number of ways to distribute the balls in each respective scenario. In the former, we have $5 \cdot 4 = 20$ ways to choose which bins end up with 3 and 5 balls; and, $\frac{20!}{3!5!(4!)^3}$ ways to choose the order of the balls. In the latter, we have $\frac{20!}{(4!)^5}$ ways to choose the order of the balls. Thus,

$$\frac{p}{q} = \frac{20 \cdot (4!)^2}{3!5!} = 16.$$

19 AMC 12A 2021/19

Let x be the least real number greater than 1 such that $\sin(x) = \sin(x^2)$, where the arguments are in degrees. What is x rounded up to the closest integer?

(A) 10 **(B)** 13 **(C)** 14 **(D)** 19 **(E)** 20

Solution. $\boxed{\text{B}}$

Clearly, the least x occurs when $x = 180 - x^2$. By plugging in $x = 12$ and $x = 13$, it is easy to see that the solution falls between these two integers; therefore, rounded up, our answer is 13.

20 AMC 12A 2021/20

For each positive integer n, let $f_1(n)$ be twice the number of positive integer divisors of n, and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$?

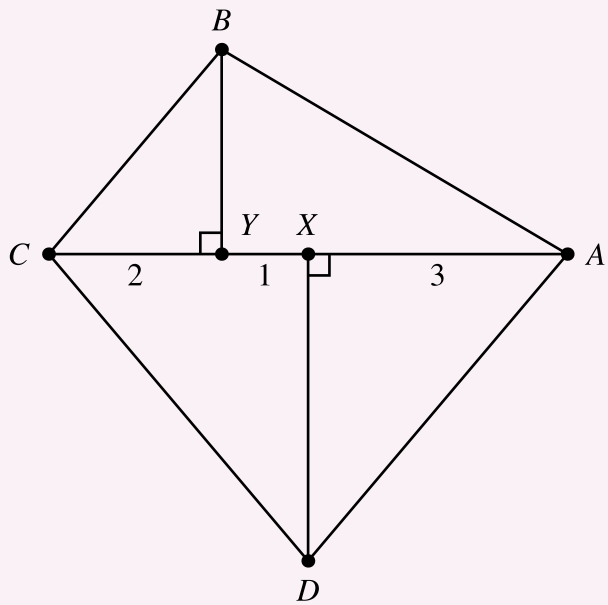
(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Solution. D

The key observation is that all numbers in the given range end up at 12 or 8, due to the fact that $f(12) = 12$ and $f(8) = 8$. By checking prime decompositions, all prime $n = p$ end up at 8; all semiprime $n = p_1 \cdot p_2$ end up at 8; all prime squares $n = p^2$ end up at 8; all $n = p_1^2 \cdot p_2$ end up at 12; all prime cubes $n = p^3$ end up at 8. This already covers the vast majority of cases $1 \leq n \leq 50$; by checking the remaining cases, we get 10 total solutions.

21 AMC 12A 2021/21

Let $ABCD$ be an isosceles trapezoid with $\overline{BC} \parallel \overline{AD}$ and $AB = CD$. Points X and Y lie on diagonal $\overline{AC}$ with X between A and Y, as shown in the figure. Suppose $\angle AXD = \angle BYC = 90^\circ$, $AX = 3$, $XY = 1$, and $YC = 2$. What is the area of $ABCD$?



(A) 15 **(B)** $5\sqrt{11}$ **(C)** $3\sqrt{35}$ **(D)** 18 **(E)** $7\sqrt{7}$

Solution. C

Using $BD = AC = 6$, we get that the area is equal to $\frac{AC}{2}(BY + XD) = 3\sqrt{BD^2 - XY^2} = 3\sqrt{35}$.

Alternatively, using similar triangles we can set $BY = 2k$ and $XD = 3k$, from which $AB = CD$ gives us $4^2 + 4k^2 = 3^2 + 9k^2 \implies k = \frac{\sqrt{35}}{5}$. This similarly gives us an area equal to $3 \cdot 5 \cdot \sqrt{35}/5 = 3\sqrt{35}$.

22 AMC 12A 2021/22

Azar and Carl play a game of tic-tac-toe. Azar places an X in one of the boxes in the 3-by-3 array of boxes, then Carl places an O in one of the remaining boxes. After that, Azar places an X in one of the remaining boxes, and so on until all 9 boxes are filled or one of the players has 3 of their symbols in a row — horizontal, vertical, or diagonal — whichever comes first, in which case that player wins the game. Suppose the players make their moves at random, rather than trying to follow a rational strategy, and that Carl wins the game when he places his third O. How many ways can the board look after the game is over?

(A) 36 **(B)** 112 **(C)** 120 **(D)** 148 **(E)** 160

Solution. $\boxed{\text{D}}$

In each of the 8 possible positions for Carl to have placed his 3 O's, there are $9 - 3 = 6$ places for Azar to have played his 3 X's, for a total of $8 \cdot \binom{6}{3}$ combinations. However, we must subtract the boards when Azar won first; this is only possible in the 6 positions when Carl's O's form a line, each giving 2 possible boards for Azar to have won first. Therefore, our final answer is $8 \cdot \binom{6}{3} - 2 \cdot 6 = 148$.

23 AMC 12A 2021/23

A quadratic polynomial $p(x)$ with real coefficients and leading coefficient 1 is called disrespectful if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

(A) $\frac{5}{16}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{8}$ **(D)** 1 **(E)** $\frac{9}{8}$

Solution. $\boxed{\text{A}}$

Let a, b be the roots of $p(x)$. We wish to maximize $a + b$ under the condition that exactly one of the two equations:

$$\begin{cases} p(x) - a = (x - a)(x - b) - a = 0 \\ p(x) - b = (x - a)(x - b) - b = 0 \end{cases}$$

has discriminant 0 (one solution). Without loss of generality, suppose the first condition satisfies this condition. Then, we get that

$$(a + b)^2 - 4(ab - a) = 0 \implies (a - b)^2 = -4a \implies b = a - 2\sqrt{-a}.$$

Because we want to maximize the value of $a + b$, we maximize

$$a + b = 2a - 2\sqrt{-a} = 2\Big(-(\sqrt{-a})^2 - \sqrt{-a}\Big) \le 2\left(\frac{1}{4}\right) = \frac{1}{2},$$

which follows from the fact that $-(\sqrt{-a} + \frac{1}{2})^2 \le 0$. We have equality when $\sqrt{-a} = -1/2$, or when $a = -1/4$; this gives $b = 3/4$, from which $\tilde{p}(1) = (1 + \frac{1}{4})(1 - \frac{3}{4}) = \frac{5}{16}$.

For completeness, we should also make sure that $p(x) - b = (x + \frac{1}{4})(x - \frac{3}{4}) - \frac{3}{4}$ has two distinct real roots (by necessity, these two roots are different than the single root in $p(x) - a = 0$ because $a \ne b$), which follows from the fact that

$$\left(\frac{1}{2}\right)^2 - 4\left(-\frac{3}{16} - \frac{3}{4}\right) > 0.$$

24 AMC 12A 2021/24

Problem 24.

Convex quadrilateral $ABCD$ has $AB = 18, \angle A = 60^\circ$, and $\overline{AB} \parallel \overline{CD}$. In some order, the lengths of the four sides form an arithmetic progression, and side $\overline{AB}$ is a side of maximum length. The length of another side is a. What is the sum of all possible values of a?

(A) 24 **(B)** 42 **(C)** 60 **(D)** 66 **(E)** 84

Solution. $\boxed{\text{E}}$

Because AB is our longest side, define $E \in AB$ such that $AECD$ is a parallelogram. We proceed with casework on the different ways to label sides BC, CD, DA with $18 - x, 18 - 2x, 18 - 3x$, for $0 \le x < 6$.

If $DC = 18 - x$, then $EB = 18 - DC = x$, and $\{AD, CB\} = \{EC, CB\} = \{18 - 2x, 18 - 3x\}$. In both cases, $\triangle ECB$ is degenerate, and $ABCD$ is a rhombus, so $a = 18$ works.

If $DC = 18 - 2x$, then $EB = 2x$, and $\{EC, CB\} = \{18 - x, 18 - 3x\}$, again giving us $a = 18$ as our only solution.

If $DC = 18 - 3x$, then $EB = 3x$, and $\{EC, CB\} = \{18 - x, 18 - 2x\}$. If $EC = 18 - x$, then by law of cosines on $\triangle ECB$ we have:

$$(18 - 2x)^2 = (18 - x)^2 + (3x)^2 - (3x)(18 - x) \implies x = 2,$$

giving us $\{AD, DC, CB\} = \{12, 14, 16\}$. If $EC = 18 - 2x$, then we similarly have:

$$(18 - x)^2 = (18 - 2x)^2 + 9x^2 - 3x(18 - 2x) \implies x = 5,$$

giving us $\{AD, DC, CB\} = \{3, 8, 13\}$.

Finally, summing over all possible a, we get that $3 \cdot 8 + 3 \cdot 14 + 18 = 84$ is our answer.

25 AMC 12A 2021/25

Let $m \geq 5$ be an odd integer, and let $D(m)$ denote the number of quadruples (a_1, a_2, a_3, a_4) of distinct integers with $1 \leq a_i \leq m$ for all i such that m divides $a_1 + a_2 + a_3 + a_4$. There is a polynomial

$$q(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

such that $D(m) = q(m)$ for all odd integers $m \geq 5$. What is c_1?

(A) -6 **(B)** -1 **(C)** 4 **(D)** 6 **(E)** 11

Solution. $\boxed{\text{E}}$

There are $m(m-1)(m-2)(m-3)$ total possible quadruples. Consider the quantity $i \equiv a_1 + a_2 + a_3 + a_4 \pmod{m}$. Intuitively, across all $0 \leq i \leq m-1$, the number of quadruples satisfying this equality is symmetric; therefore, we conjecture that the number of quadruples satisfying $i = 0$ is precisely

$$D(m) = \frac{1}{m} \cdot m(m-1)(m-2)(m-3) \implies c_1 = 11.$$

Indeed, consider the family of bijections $(a_1, a_2, a_3, a_4) \mapsto (a_1 + j, a_2 + j, a_3 + j, a_4 + j)$ over all $1 \leq j \leq m-1$ (each quadruple taken modulo m), which maps modulo m the i-th residue class to the $(i + 4j)$-th. Because m is odd, these $m - 1$ residue classes are precisely those not equivalent to i; therefore, the number of valid quadruples satisfying each residue is equal, and our conjecture holds.

Remark. This fails when m is even because $i+4j$ only covers those residue classes with the same parity as i; in particular, $m \equiv 0 \pmod{4}$ and $m \equiv 2 \pmod{4}$ yield two distinct polynomials different than $D(m)$. This can be explained intuitively through cases like $(a_1, a_2, a_3, a_4) = (a, a, b, b)$, which are not symmetric modulo even m because they can only ever produce even residues.