2021 AMC 12A SOLUTIONS

Stevenson Math Team

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Problem 1.

What is the value of
$$2^{1+2+3} - (2^1 + 2^2 + 2^3)?$$
(A) 0 **(B)** 50 **(C)** 52 **(D)** 54 **(E)** 57

Problem 2.

Under what conditions is $\sqrt{a^2 + b^2} = a + b$ true, where a and b are real numbers?

(A) It is never true.
(B) It is true if and only if $ab = 0$.
(C) It is true if and only if $a + b \geq 0$.
(D) It is true if and only if $ab = 0$ and $a + b \geq 0$.
(E) It is always true.

Problem 3.

The sum of two natural numbers is 17,402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

(A) 10,272 **(B)** 11,700 **(C)** 13,362 **(D)** 14,238 **(E)** 15,426

Problem 4.

Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that

- all of his happy snakes can add,
- none of his purple snakes can subtract, and
- all of his snakes that can't subtract also can't add.

Which of these conclusions can be drawn about Tom's snakes?

(A) Purple snakes can add.
(B) Purple snakes are happy.
(C) Snakes that can add are purple.
(D) Happy snakes are not purple.
(E) Happy snakes can't subtract.

Problem 5.

When a student multiplied the number 66 by the repeating decimal,

$$\underline{1}.\underline{a}\,\underline{b}\,\underline{a}\,\underline{b}\ldots = \underline{1}.\overline{\underline{a}\,\underline{b}},$$

where a and b are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{a}\,\underline{b}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit number $\underline{a}\,\underline{b}$?

(A) 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

Problem 6.

A deck of cards has only red cards and black cards. The probability of a randomly chosen card being red is $\frac{1}{3}$. When 4 black cards are added to the deck, the probability of choosing red becomes $\frac{1}{4}$. How many cards were in the deck originally?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

Problem 7.

What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers x and y?

(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

Problem 8.

A sequence of numbers is defined by $D_0=0$, $D_1=0$, $D_2=1$, and $D_n=D_{n-1}+D_{n-3}$ for $n\geq 3$. What are the parities (evenness or oddness) of the triple of numbers $(D_{2021}, D_{2022}, D_{2023})$, where E denotes even and O denotes odd?

(A) (O,E,O) **(B)** (E,E,O) **(C)** (E,O,E) **(D)** (O,O,E) **(E)** (O,O,O)

Problem 9.

Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$
(D) $3^{128}+2^{128}$ **(E)** 5^{127}

Problem 10.

Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?

(A) $1:1$ **(B)** $47:43$ **(C)** $2:1$ **(D)** $40:13$ **(E)** $4:1$

Problem 11.

A laser is placed at the point $(3,5)$. The laser beam travels in a straight line. Larry wants the beam to hit and bounce off the y-axis, then hit and bounce off the x-axis, then hit the point $(7,5)$. What is the total distance the beam will travel along this path?

(A) $2\sqrt{10}$ **(B)** $5\sqrt{2}$ **(C)** $10\sqrt{2}$ **(D)** $15\sqrt{2}$ **(E)** $10\sqrt{5}$

Problem 12.

All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B?
(A) -88 **(B)** -80 **(C)** -64 **(D)** -41 **(E)** -40

Problem 13.

Of the following complex numbers z, which one has the property that z^5 has the greatest real part?

(A) -2 **(B)** $-\sqrt{3}+i$ **(C)** $-\sqrt{2}+\sqrt{2}i$ **(D)** $-1+\sqrt{3}i$ **(E)** $2i$

Problem 14.

What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right)\left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

(A) 21 **(B)** $100\log_5 3$ **(C)** $200\log_3 5$ **(D)** $2,200$ **(E)** $21,000$

Problem 15.

A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that could be selected. What is the remainder when N is divided by 100?

(A) 47 **(B)** 48 **(C)** 83 **(D)** 95 **(E)** 96

Problem 16.

In the following list of numbers, the integer n appears n times in the list for $1 \le n \le 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

(A) 100.5 **(B)** 134 **(C)** 142 **(D)** 150.5 **(E)** 167

Problem 17.

Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

Problem 18.

Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

Problem 19.

How many solutions does the equation $\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$ have in the closed interval $[0, \pi]$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Problem 20.

Suppose that on a parabola with vertex V and focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

(A) 13 **(B)** $\frac{14}{3}$ **(C)** $\frac{41}{3}$ **(D)** 14 **(E)** $\frac{43}{3}$

Problem 21.

The five solutions to the equation

$$(z-1)(z^2+2z+4)(z^2+4z+6) = 0$$

may be written in the form $x_k + y_k i$ for $1 \le k \le 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5). The eccentricity of $\mathcal{E}$ can be written in the form $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. What is $m + n$? (Recall that the *eccentricity* of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of $\mathcal{E}$ and $2c$ is the distance between its two foci.)

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

Problem 22.

Suppose that the root of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where the angles are in radians. What is abc ?

(A) $-\frac{3}{49}$ **(B)** $-\frac{1}{28}$ **(C)** $\frac{\sqrt[3]{7}}{64}$ **(D)** $\frac{1}{32}$ **(E)** $\frac{1}{28}$

Problem 23.

Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop–up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

(A) $\frac{9}{16}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{25}{32}$ **(E)** $\frac{13}{16}$

Problem 24.

Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, then the area of $\triangle PQR$ is $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. What is $a + b + c$?

(A) 110 **(B)** 114 **(C)** 118 **(D)** 122 **(E)** 126

Problem 25.

Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1$, $d(2) = 2$, and $d(12) = 6$. (This function is known as the *divisor function*.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

1 AMC 12A 2021/1

What is the value of

$$2^{1+2+3} - (2^1 + 2^2 + 2^3)?$$

(A) 0 **(B)** 50 **(C)** 52 **(D)** 54 **(E)** 57

Solution. $\boxed{\text{B}}$

$2^6 - (2 + 4 + 8) = 50.$

2 AMC 12A 2021/2

Under what conditions is $\sqrt{a^2+b^2}=a+b$ true, where a and b are real numbers?

(A) It is never true.
(B) It is true if and only if $ab=0$.
(C) It is true if and only if $a+b\geq 0$.
(D) It is true if and only if $ab=0$ and $a+b\geq 0$.
(E) It is always true.

Solution. $\boxed{\text{D}}$

Clearly, $a+b$ must be nonnegative because of the square root. Also, squaring both sides gives $ab=0$, so our answer is D.

3 AMC 12A 2021/3

The sum of two natural numbers is $17,402$. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

(A) $10,272$ **(B)** $11,700$ **(C)** $13,362$ **(D)** $14,238$ **(E)** $15,426$

Solution. $\boxed{\text{D}}$

if we let a be the value of the smaller number, then the problem tells us

$$a + 10a = 17402 \implies a = 1582.$$

Therefore, our two numbers are 15820 and 1582, giving us D as our final answer.

4 AMC 12A 2021/4

Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that

- all of his happy snakes can add,
- none of his purple snakes can subtract, and
- all of his snakes that can't subtract also can't add.

Which of these conclusions can be drawn about Tom's snakes?

(A) Purple snakes can add.
(B) Purple snakes are happy.
(C) Snakes that can add are purple.
(D) Happy snakes are not purple.
(E) Happy snakes can't subtract.

Solution. $\boxed{\text{D}}$

The third condition tells us that purple snakes cannot add nor subtract. This eliminates choice A and C. We also know that happy snakes can add so they must not be purple, eliminating B. Also, if happy snakes could not subtract, then the third condition would tell us that they also could not add, contradicting the first condition, so we can also eliminate option B, leaving us with D as our final answer.

5 AMC 12A 2021/5

When a student multiplied the number 66 by the repeating decimal,

$$\underline{1}.\underline{a}\,\underline{b}\,\underline{a}\,\underline{b}\ldots = \underline{1}.\overline{\underline{a}\,\underline{b}},$$

where a and b are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{a}\,\underline{b}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit number $\underline{a}\,\underline{b}$?

(A) 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

Solution. $\boxed{\text{E}}$

Note that by infinite geometric series, we have

$$\underline{1}.\overline{\underline{a}\,\underline{b}} = 1 + \frac{\underline{a}\,\underline{b}}{99},$$

and we also know that

$$\underline{1}.\underline{a}\,\underline{b} = 1 + \frac{\underline{a}\,\underline{b}}{100}.$$

Therefore, the given problem reads:

$$\left(1 + \frac{\underline{a}\,\underline{b}}{99}\right) \cdot 66 + 0.5 = 1 + \left(\frac{\underline{a}\,\underline{b}}{100}\right) \cdot 66,$$

and solving for $\underline{a}\,\underline{b}$ gives us

$$\underline{a}\,\underline{b} = 75.$$

6 AMC 12A 2021/6

A deck of cards has only red cards and black cards. The probability of a randomly chosen card being red is $\frac{1}{3}$. When 4 black cards are added to the deck, the probability of choosing red becomes $\frac{1}{4}$. How many cards were in the deck originally?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

Solution. $\boxed{\text{C}}$

Let r denote the number of red cards originally in the deck, so that the deck initially had r red cards and $2r$ black cards. The stipulation in the problem then tells us that

$$\frac{r}{3r+4} = \frac{1}{4},$$

whence $r = 4$ and the total number of cards in the original deck was 12.

7 AMC 12A 2021/7

What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers x and y?

(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

Solution. $\boxed{\text{D}}$

$(xy-1)^2+(x+y)^2=(xy)^2+x^2+y^2+1=(x^2+1)(y^2+1)\geq 1.$

8 AMC 12A 2021/8

A sequence of numbers is defined by $D_0 = 0$, $D_1 = 0$, $D_2 = 1$, and $D_n = D_{n-1} + D_{n-3}$ for $n \geq 3$. What are the parities (evenness or oddness) of the triple of numbers $(D_{2021}, D_{2022}, D_{2023})$, where E denotes even and O denotes odd?

(A) (O,E,O) **(B)** (E,E,O) **(C)** (E,O,E) **(D)** (O,O,E) **(E)** (O,O,O)

Solution. $\boxed{\text{C}}$

Note that the first few terms of the sequence are $E, E, O, O, O, E, O, E, E, O, \ldots$. Therefeore, because each new element in the sequence only relies on the previous three terms, and $D_0 = D_7 = E$, $D_1 = D_8 = E$, and $D_2 = D_9 = O$, the parities of our sequence repeats itself every 7 terms. Noting that $2021 \equiv 5 \mod 7$, then the parity of the triple $(D_{2021}, D_{2022}, D_{2023})$ is equal to the parity of the triple $(D_5, D_6, D_7) = (E, O, E)$.

9 AMC 12A 2021/9

Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$
(D) $3^{128}+2^{128}$ **(E)** 5^{127}

Solution. $\boxed{\text{C}}$

Let S denote the value of the given product. Then,

$$\begin{aligned}(3-2)S &= (3-2)(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64}) \\ &= (3^2-2^2)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64}) \\ &\ \vdots \\ &= 3^{128}-2^{128},\end{aligned}$$

so $S = 3^{128}-2^{128}$.

10 AMC 12A 2021/10

Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?

(A) 1 : 1 **(B)** 47 : 43 **(C)** 2 : 1 **(D)** 40 : 13 **(E)** 4 : 1

Solution. $\boxed{\text{E}}$

Note that both the initial and final volumes of water in each cone are the same, and therefore the scale factor between each initial and final cone between the two cones is also the same. Because the ratio of the initial radii in each cone was 1 : 2, then we must have that the ratio of their heights is 4 : 1 because we are given that they start with the same volume. Finally, because both cones increased in size by the same scale factor after the marble is dropped into them, then the ratio of the increase in their heights is also 4 : 1.

(if this isn't convincing, try testing out the change in heights with more concrete numbers, e.g letting the two heights be $4x$ and x and seeing how dropping in the marble affects the final heights of the two cones).

11 AMC 12A 2021/11

A laser is placed at the point $(3,5)$. The laser beam travels in a straight line. Larry wants the beam to hit and bounce off the y-axis, then hit and bounce off the x-axis, then hit the point $(7,5)$. What is the total distance the beam will travel along this path?

(A) $2\sqrt{10}$ **(B)** $5\sqrt{2}$ **(C)** $10\sqrt{2}$ **(D)** $15\sqrt{2}$ **(E)** $10\sqrt{5}$

Solution. $\boxed{\text{C}}$

By the law of reflections, the problem is equivalent to finding the distance between $(3,5)$ and $(-7,-5)$, the point obtained after $(7,5)$ is reflected across the x-axis, and then y-axis, respectively. This gives us $10\sqrt{2}$ as our final answer.

12 AMC 12A 2021/12

All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B?
(A) -88 **(B)** -80 **(C)** -64 **(D)** -41 **(E)** -40

Solution. $\boxed{\text{A}}$

By Vieta's, the sum of the roots of of our polynomial is 10, and their product is 2^4. Because we are given that every root is a positive integer, we thus know that the 6 roots must be

$$2, 2, 2, 2, 1, 1.$$

Finally, by Vieta's again we must have that $-B$ is equal to the sum of all $\binom{6}{3} = 20$ products between any three of the roots. 4 of these products are equal to $2^3 = 8$, 12 of these products are equal to $2^2 = 4$, and the remaining 4 of these products are equal to $2^1 = 2$, giving us $B = -88$ as our final answer.

13 AMC 12A 2021/13

Of the following complex numbers z, which one has the property that z^5 has the greatest real part?

(A) -2 **(B)** $-\sqrt{3}+i$ **(C)** $-\sqrt{2}+\sqrt{2}i$ **(D)** $-1+\sqrt{3}i$ **(E)** $2i$

Solution. $\boxed{\text{B}}$

First note that the magnitude (i.e, distance from the origin in the complex plane) of each answer choice is 2; therefore, by De Moivre's Theorem, it suffices to find the 5 times the argument of each answer choice, and see which of these numbers is "closest" to any multiple of 360.

The arguments of each answer choice, in order, are

$$180^\circ, 150^\circ, 135^\circ, 120^\circ, 90^\circ.$$

Multiplying each of these arguments by 5 and taking mod 360 then gives us

$$180^\circ, 30^\circ, -45^\circ, -120^\circ, 90^\circ,$$

from which it is clear that option B is our answer.

14 AMC 12A 2021/14

What is the value of
$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right)\left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

(A) 21 **(B)** $100\log_5 3$ **(C)** $200\log_3 5$ **(D)** $2,200$ **(E)** $21,000$

Solution. $\boxed{\text{E}}$

For all k, we have that
$$\log_{5^k} 3^{k^2} = \frac{k^2\log 3}{k\log 5} = k\log_5 3,$$
and
$$\log_{9^k} 25^k = \frac{2k\log 5}{2k\log 3} = \log_3 5,$$
therefore our desired product is equal to
$$\begin{aligned}\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right)\left(\sum_{k=1}^{100} \log_{9^k} 25^k\right) &= \left(\sum_{k=1}^{20} k\log_5 3\right)\left(\sum_{k=1}^{100} \log_3 5\right)\\ &= 210\log_5 3\cdot 100\log_3 5\\ &= 21,000.\end{aligned}$$

15 AMC 12A 2021/15

A choir director must select a group of singers from among his 6 tenors and 8 basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of 4, and the group must have at least one singer. Let N be the number of groups that could be selected. What is the remainder when N is divided by 100?

(A) 47 **(B)** 48 **(C)** 83 **(D)** 95 **(E)** 96

Solution. $\boxed{\text{D}}$

We proceed with casework on the number of tenors.

If there are 0 or 4 tenors, then there must be 0, 4 or 8 bases, for a total of $\binom{6}{0}\binom{8}{4} + \binom{6}{0}\binom{8}{8} + \binom{6}{4}\binom{8}{0} + \binom{6}{4}\binom{8}{4} + \binom{6}{4}\binom{8}{8}$ possible groups for this case (excluding the case where there are 0 tenors and 0 bases).

If there are 1 or 5 tenors, then there must be 1 or 5 bases, for a total of $\binom{6}{1}\binom{8}{1} + \binom{6}{1}\binom{8}{5} + \binom{6}{5}\binom{8}{1} + \binom{6}{5}\binom{8}{5}$ possible groups for this case.

If there are 2 or 6 tenors, then there must be 2 or 6 bases, for a total of $\binom{6}{2}\binom{8}{2} + \binom{6}{2}\binom{8}{6} + \binom{6}{6}\binom{8}{2} + \binom{6}{6}\binom{8}{6}$ possible groups for this case.

If there are 3 tenors, then there must be 3 or 7 bases, for a total of $\binom{6}{3}\binom{8}{3} + \binom{6}{3}\binom{8}{7}$ possible groups for this case.

Summing over all four cases (by e.g Vandermonde's, or straight bashing), we thus have that the total number of groups is equal to 4095.

16 AMC 12A 2021/16

In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

(A) 100.5 **(B)** 134 **(C)** 142 **(D)** 150.5 **(E)** 167

Solution. $\boxed{\text{C}}$

We want to find the least k such that $1 + \ldots + k \geq \frac{1}{2} \cdot \frac{200 \cdot 201}{2}$ (half of the total amount of numbers in the list). This simplifies down to finding the least k such that

$$k(k+1) \geq 201 \cdot 100.$$

From here, there are many ways to proceed, but one relatively quick way to solve for k without directly plugging in the answer choices is to note that because $201 \approx 2 \cdot 100$, then $k \approx \sqrt{2} \cdot 100 \approx 141$, from which it is easy to verify that answer choice C works.

17 AMC 12A 2021/17

Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

Solution. $\boxed{\text{D}}$

If we let M be the midpoint of AB, then $\triangle BMP \sim \triangle BAD$ gives us that MP perpendicularly bisects BD. Also, because $\triangle BCD$ is isosceles, DP also perpendicularly bisects BD, therefore $MDCB$ is a rhombus. From $AM = MB$ it then follows that $AD \parallel CP$, giving us $\triangle ADO \sim \triangle CPO$. Because $CP = \frac{1}{2}AD$ (from e.g $MP = \frac{1}{2}AD$), we thus have that $DO = 22$ and therefore from $\triangle DCP$ we get

$$AD = 2\sqrt{43^2 - 33^2} = 4\sqrt{190}.$$

18 AMC 12A 2021/18

Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

Solution. $\boxed{\text{E}}$

$f(25) = f(\frac{25}{11}) + f(11) \implies f(\frac{25}{11}) = 2f(5) - f(11) = -1.$

19 AMC 12A 2021/19

How many solutions does the equation $\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$ have in the closed interval $[0, \pi]$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution. $\boxed{\text{C}}$

Using the well known identity $\sin(x) = \cos(\frac{\pi}{2} - x)$, the given equation simplifies down to

$$\cos(\frac{\pi}{2} - \frac{\pi}{2}\cos x) = \cos(\frac{\pi}{2}\sin x).$$

Because our interval is restricted, we can set the inside of both cosines equal to get that $1 - \cos x = \sin x$. Either by graphing or more algebraic manipulation, from here it is clear that there are only two solutions in the given interval.

20 AMC 12A 2021/20

Suppose that on a parabola with vertex V and focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

(A) 13 **(B)** $\frac{40}{3}$ **(C)** $\frac{41}{3}$ **(D)** 14 **(E)** $\frac{43}{3}$

Solution. $\boxed{\text{B}}$

Noting that there is only one unique scaling factor for any parabola, we can without loss of generality let our parabola take on the equation

$$y = ax^2,$$

for positive a. By definition, $V = (0, 0)$ and $F = (0, \frac{1}{4a})$. Letting $A = (x, ax^2)$, by the distance formula we have

$$AV = 21 = \sqrt{x^2 + (ax^2)^2}$$
$$AF = 20 = \sqrt{x^2 + (ax^2 - \frac{1}{4a})^2}.$$

Squaring both equations and subtracting gives

$$41 = \frac{x^2}{2} - \frac{1}{16a^2} \implies x^2 = 82 + \frac{1}{16a^2}.$$

By the definition of a parabola, we also know that the distance from A to its directrix $y = -\frac{1}{4a}$ is also 20; therefore, we also have

$$ax^2 + \frac{1}{4a} = 20,$$

from which plugging in our previous expression for x^2 gives

$$82a + \frac{3}{8a} = 20.$$

Because we want to find the sum of all possible values of $FV = \frac{1}{4a}$, let $k = \frac{1}{4a}$, and substituting k into our above equation gives us the quadratic

$$3k^2 - 40k + 41,$$

from which it follows from Vieta's that our final answer is $\frac{40}{3}$ (it is easy to verify that both solutions to the quadratic work).

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The five solutions to the equation

$$(z-1)(z^2+2z+4)(z^2+4z+6)=0$$

may be written in the form $x_k+y_k i$ for $1 \le k \le 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4), (x_5,y_5). The eccentricity of $\mathcal{E}$ can be written in the form $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers. What is $m+n$? (Recall that the *eccentricity* of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of $\mathcal{E}$ and $2c$ is the distance between its two foci.)

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

Solution. $\boxed{\text{A}}$

By the quadratic equation, the five roots of the equation are 1, $-1 \pm \sqrt{3}i$, and $-2 \pm \sqrt{2}i$. We can now manually find the equation of our ellipse; because the roots are symmetrical with respect to the x-axis, we set

$$\mathcal{E} := \frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Plugging in each of our roots gives us the three equations:

$$\begin{aligned}(1-h)^2 &= a^2\\ b^2(1+h)^2 &= a^2(b^2-3)\\ b^2(2+h)^2 &= a^2(b^2-2).\end{aligned}$$

Substituting our first equation into the latter two, and isolating a^2 gives us

$$\begin{aligned}-3a^2 &= b^2(4h)\\ -2a^2 &= b^2(3+6h),\end{aligned}$$

whence dividing these two equations gives us $h=-\frac{9}{10}$. Thus, $\frac{b^2}{a^2}=-\frac{3}{4h}=\frac{5}{6}$ and our desired eccentricity is $\sqrt{\frac{1}{6}}$.

22 AMC 12A 2021/22

Suppose that the root of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where the angles are in radians. What is abc ?

(A) $-\frac{3}{49}$ **(B)** $-\frac{1}{28}$ **(C)** $\frac{\sqrt[3]{7}}{64}$ **(D)** $\frac{1}{32}$ **(E)** $\frac{1}{28}$

Solution. $\boxed{\text{D}}$

Let $\theta = \frac{2\pi}{7}$. By roots of unity, it follows that

$$\sum_{k=1}^{6} \cos k\theta = -1 \implies \cos(\theta) + \cos(2\theta) + \cos(3\theta) = -\frac{1}{2}.$$

Now let $\cos(\theta) = x$. Note that we then have $\cos(2\theta) = 2x^2 - 1$, and $\cos(3\theta) = 4x^3 - 3x$, therefore

$$x + 2x^2 - 1 + 4x^3 - 3x = -\frac{1}{2} \implies x^3 + \frac{x^2}{2} - \frac{x}{2} - \frac{1}{8} = 0,$$

so our final answer is $\frac{1}{32}$.

23 AMC 12A 2021/23

Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop–up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

(A) $\frac{9}{16}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{25}{32}$ **(E)** $\frac{13}{16}$

Solution. $\boxed{\text{D}}$

Define (k, L) to be the *state* of Frieda, where k is the number of moves she has already made and $L \in \{c, e\}$ is her position on the board (either a center square or edge square). Let $p_{(k,L)}$ be the probability that Frieda is successful from the state (k, L).

Clearly, $p_{(0,c)} = p_{(1,e)}$. From the state $(1, e)$, there is a $\frac{1}{2}$ probability that she wins on her next move, and a $\frac{1}{4}$ probability that she transitions to the states $(2, e)$, $(2, c)$, respectively, so $p_{(1,e)} = \frac{1}{2} + \frac{1}{4}p_{(2,e)} + \frac{1}{4}p_{(2,c)}$. By the same line of reasoning, $p_{(2,c)} = p_{(3,e)}$ and $p_{(2,e)} = \frac{1}{2} + \frac{1}{4}p_{(3,e)} + \frac{1}{4}p_{(3,c)}$. Finally, because she stops hopping after four moves, $p_{(3,c)} = 0$ and $p_{(3,e)} = \frac{1}{2}$. Substituting in these values and backsolving gives $p_{(0,c)} = \frac{25}{32}$.

24 AMC 12A 2021/24

Problem 24.

Semicircle Γ has diameter $\overline{AB}$ of length 14. Circle Ω lies tangent to $\overline{AB}$ at a point P and intersects Γ at points Q and R. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, then the area of $\triangle PQR$ is $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. What is $a + b + c$?

(A) 110 **(B)** 114 **(C)** 118 **(D)** 122 **(E)** 126

Solution. $\boxed{\text{D}}$

Denote O the center of Γ, U the center of Ω, and M the midpoint of QR. By the extended law of sines, the radius of Ω is equal to

$$R = \frac{3\sqrt{3}}{2\sin 60^\circ} = 3.$$

Therefore, from $\triangle UMR$ we get $UM = \frac{3}{2}$. Note also that we must have $\overline{OUM}$ collinear because both OM and UM perpendicularly bisect QR, thus we can use the pythagorean theorem on $\triangle OMR$ to give us $OM = \frac{13}{2}$. We now have $UO = OM - UM = 5$, and because $PU = 3$, we must have $PO = 4$. Finally, if X is the foot of the altitude from P in $\triangle POU$, then we know $UX = \frac{9}{5}$, giving us

$$[PQR] = \frac{QR \cdot MX}{2} = \frac{99\sqrt{3}}{20}.$$

25 AMC 12A 2021/25

Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1$, $d(2) = 2$, and $d(12) = 6$. (This function is known as the *divisor function*.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Solution. $\boxed{\text{E}}$

Note that f is a multiplicative function as $f(mn) = f(m)f(n)$ for coprime m and n. This means that we can just maximize f on powers of prime numbers and then multiply our answers. Effectively, for each prime p we need to find k such that $f(p^k) = \frac{k+1}{\sqrt[3]{p^k}}$ is maximized. Bashing out primes starting from $p = 2$, we get that the function is maximized at $2^3, 3^2, 5^1, 7^1$ and no other primes as they maximize at $k = 0$. From here, $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, thus the answer is $\boxed{E}$.